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A new generalization of semiregular rings

Tayyebeh Amouzegar

Abstract. A ring R is called *ν-semiregular* if for every semisimple principal right ideal aR of R, there exists $e^2 = e \in aR$ such that $(1-e)a \in J(R)$. The class of right ν-semiregular rings contains all semiregular rings. Some properties of these rings are studied and some results about semiregular rings are extended.

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1. Introduction

Semiregular rings were introduced by Nicholson in 1976. These rings constitute the class of rings that posses beautiful homological and non homological properties. Following [3], a ring R is called a *semiregular* ring if for each $a \in R$, there exists $e^2 = e \in aR$ such that $(1-e)a \in J(R)$. Semiregular rings and their generalizations have been studied by many authors (see [1, 2, 5, 6, 7]). In this note, we define ν-semiregular rings, as a generalization of semiregular rings. A ring R is called *ν-semiregular* if for every semisimple principal right ideal aR of R, there exists $e^2 = e \in aR$ such that $(1-e)a \in J(R)$. Clearly, any semiregular ring is ν-semiregular but ν-semiregular rings need not be semiregular (see Example 4.1). In this paper our aim is to generalize some corresponding known results on semiregular rings.

In Section 2, we introduce the concept of the θ equivalence relation on the set of right ideals of a ring. We say right ideals I, I' of R are *θ equivalent*, $I\theta I'$, if and only if $\frac{I+I'}{I} \subseteq \frac{J(R)+I}{I}$ and $\frac{I+I'}{I'} \subseteq \frac{J(R)+I'}{I'}$. We investigate some basic properties of the θ relation.

In Section 3, we use the θ relation to give a new characterization of semiregular rings.

In Section 4, a characterization of ν-semiregular rings is given. We examine when direct sum of ν-semiregular rings is ν-semiregular. We give some sufficient conditions under which a ν-semiregular ring is semiregular.

Throughout this paper R will denote an associative ring with identity, M a unitary right R-module. We will use the notation $N \ll M$ to indicate that N is small in M (i.e. $\forall L \lneq M, L + N \neq M$).

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2. The θ relation

Definition 2.1. Any right ideals I, I' of R are θ equivalent, $I\theta I'$, if and only if $\frac{I+I'}{I} \subseteq \frac{J(R)+I}{I}$ and $\frac{I+I'}{I'} \subseteq \frac{J(R)+I'}{I'}$.

Lemma 2.1. *θ is an equivalence relation.*

Proof. It is clear that the reflexive and symmetric properties satisfy. For transitivity, suppose $A\theta B$ and $B\theta C$. So

$$\begin{array}{ccc} \frac{A+B}{A} \subseteq \frac{J(R)+A}{A} & \text{and} & \frac{A+B}{B} \subseteq \frac{J(R)+B}{B} \\ \frac{B+C}{B} \subseteq \frac{J(R)+B}{B} & \text{and} & \frac{B+C}{C} \subseteq \frac{J(R)+C}{C}. \end{array}$$

So

$$\begin{array}{ccc} A+B \subseteq J(R)+A & \text{and} & A+B \subseteq J(R)+B \\ B+C \subseteq J(R)+B & \text{and} & B+C \subseteq J(R)+C. \end{array}$$

It is easy to see that $A+C \subseteq J(R)+A$ and $A+C \subseteq J(R)+C$. Hence $A\theta C$. □

Note that the zero ideal is θ equivalent to any right ideal contained in $J(R)$. Also, if $R = \mathbb{Z}$ then $m\mathbb{Z}\theta n\mathbb{Z}$ if and only if m and n are divisible by the same primes.

Theorem 2.2. *Let A and B be right ideals of R. The following are equivalent:*

(1) *$A\theta B$.*

(2) *$\frac{A+B}{A} \ll \frac{R}{A}$ and $\frac{A+B}{B} \ll \frac{R}{B}$.*

(3) *For every right ideal I of R such that $A+B+I = R$ then $A+I = M$ and $B+I = R$.*

(4) *If $H \leq R$ with $A+H = R$ then $B+H = R$, and if $K \leq R$ with $B+K = R$ then $A+K = R$.*

Proof. (1) ⇔ (2) It is clear.

(2) ⇒ (3) Let $I \leq R$ such that $A+B+I = R$. Then

$$\frac{A+B}{B} + \frac{B+I}{B} = \frac{R}{B} \Rightarrow \frac{B+I}{B} = \frac{R}{B} \Rightarrow B+I = R.$$

Similarly, $A+I = R$.

(3) ⇔ (4) Let $H \leq R$ such that $A+H = R$. Then $A+H+B = M$. By (3), $B+H = R$. Let $K \leq R$ such that $B+K = R$. Then $A+B+K = R$. By (3), $A+K = R$. Conversely, suppose that $A+(B+I) = R$. So $B+(B+I) = R$. Thus $B+I = R$. Similarly, $A+I = R$.

(3) ⇒ (2) Let $\frac{X}{B} \leq \frac{R}{B}$ such that $\frac{A+B}{B} + \frac{X}{B} = \frac{R}{B}$. Then $A+B+X = R$. Hence $B+X = X = R$ since $B \subseteq X$. Thus $\frac{A+B}{B} \ll \frac{R}{B}$. Similarly, $\frac{A+B}{A} \ll \frac{R}{A}$. □

Corollary 2.3. *Let $A, B \leq R$ such that $A \subseteq B+I$ and $B \subseteq A+I'$, where $I, I' \ll R$. Then $A\theta B$.*

Proof. Let $A+B+H = R$, for some $H \leq R$. Then $(B+I)+B+H = R$. So $B+I+H = R$. Thus $B+H = R$. Similarly, $A+H = R$. □

Note that there are rings R with $H, A, B \leq R$ such that $R = A+H = B+H$, but A is not θ related to B. For example, take $R = \mathbb{Z}$, $H = 3\mathbb{Z}$, $A = 2\mathbb{Z}$ and $B = 5\mathbb{Z}$.

Proposition 2.4. *Let A_1, A_2, B_1, B_2 be right ideals of R such that $A_1\theta B_1$ and $A_2\theta B_2$. Then $(A_1+A_2)\theta(B_1+B_2)$ and $(A_1+Y_2)\theta(B_1+A_2)$.*

Proof. Let $H \leq R$ such that $A_1+A_2+B_1+B_2+H = R$. Then $A_2+B_1+B_2+H = R$ and $A_1 + A_2 + B_2 + H = R$, since $A_1\theta B_1$. Moreover, $B_1 + B_2 + H = R$ and $A_1 + A_2 + H = R$, because $A_2\theta B_2$. By Theorem 2.2, $(A_1 + A_2)\theta(B_1 + B_2)$. From Lemma 2.1, we have $(A_1 + Y_2)\theta(B_1 + A_2)$. □

Theorem 2.5. *Let A, B be right ideals of R such that $A\theta B$. Then*

(1) *$A \ll R$ if and only if $B \ll R$.*

(2) *If A is a direct summand of R and B a principal right ideal of R, then A is also principal.*

(3) *A has a (weak) supplement C in R if and only if C is a (weak) supplement for B.*

Proof. (1) (⇒) Suppose that $A \ll R$. Let $H \leq R$ such that $B + H = M$. Then $A+B+H = R$. By Theorem 2.2, $A+H = R$. Since $A \ll R$, $H = R$. Hence $B \ll R$.

(⇐) It is clear because θ is symmetric by Lemma 2.1.

(2) Assume that $R = A\oplus A'$ for some $A' \leq R$. By Theorem 2.2, $R = B+A'$. Since $\frac{B+A'}{A'} = \frac{R}{A'} \cong A$, A is principal.

(3) Suppose that C is a supplement for A. Then $R = A + C = A + B + C$. By Theorem 2.2, $B+C = R$. Assume that $H \subseteq C$ and $B+H = R$. Then $A+B+H = R$. By Theorem 2.2, $A + H = R$. By the minimality of C, $H = C$. Hence C is a supplement for B. The converse is true because θ is symmetric (Lemma 2.1).

Now suppose that C is a weak supplement for A. Then $A+C = R$ and $A\cap C \ll R$. By Theorem 2.2, $B + C = R$. Let us to show that $B \cap C \ll R$. Let $H \leq R$ such that $B \cap C + H = R$. Since $B \cap C \subseteq B$, $B + H = R$ and $C + H = R$. By Theorem 2.2, $A + H = R$. Since $B \cap C \subseteq C$, $C = C \cap M = (B \cap C) + (C \cap H)$. Then

$$R = B + C = B + B \cap C + C \cap H = B + C \cap H.$$

Hence $A + B + C \cap H = R$. By Theorem 2.2, $A + C \cap H = R$. Hence $H = H \cap R = H\cap(C\cap H+A) = (C\cap H)+A\cap H$. Thus $R = C+H = (B\cap C)+(C\cap H)+(A\cap H) \subseteq C + A \cap H \subseteq R$. So $R = C + (A \cap H)$ and hence $A = A \cap R = A \cap ((A \cap H) + C) = A\cap H+A\cap C$. Since $A\cap C \ll R$, $R = A+H = A\cap C+A\cap H+H = A\cap H+H = H$. Therefore $B \cap C \ll R$. The converse holds by the symmetry of the θ relation. □

Proposition 2.6. *Let $R = R_1 \oplus R_2$ and $A, B \leq R_1$. Then $A\theta B$ in R if and only if $A\theta B$ in R_1.*

Proof. (⇒) Let $A\theta B$ in R and $A + B + I = R_1$ for some right ideal I of R. We want to show $A+I = R_1$ and $B+I = R_1$. Since $A\theta B$ in R, $R = R_1 \oplus R_2 = A+B+I+R_2$ implies $A + I + R_2 = R$ and $B + I + R_2 = R$. So $A + I = R_1$ and $B + I = R_1$. From Theorem 2.2, we get $A\theta B$ in R_1.

(⇐) Let $A\theta B$ in R_1. Then $\frac{A+B}{A} \subseteq \frac{J(R_1)+A}{A}$ implies $\frac{A+B}{A} \subseteq \frac{J(R)+A}{A}$. Similarly, $\frac{A+B}{B} \subseteq \frac{J(R)+A}{B}$. □

3. Semiregular rings

Recall that an element $a \in R$ is *von Neumann regular* if $a \in aRa$. A ring R is called *von Neumann regular* if, for any $a \in R$, a is von Neumann regular.

Lemma 3.1. *The following conditions are equivalent for an element a of a ring R:*

(1) *There exists $e^2 = e \in aR$ such that $(1 - e)a \in J(R)$.*

(2) *There exists* $e^2 = e \in Ra$ *such that* $a(1-e) \in J(R)$.
(3) *There exists a von Neumann regular element* $b \in R$ *with* $a - b \in J(R)$.
(4) *There exist two right ideals* A *and* B *of* R *such that* $aR = A \oplus B$, *where* A *is a direct summand of* R *and* B *is small in* R.

Proof. By [3, Lemma 2.1]. □

Let K and N be submodules of an R-module M. K is called a *supplement* of N in M if $M = K + N$ and K is minimal with respect to this property, or equivalently, $M = K + N$ and $K \cap N \ll K$ [4].

Lemma 3.2. *A ring* R *is semiregular if and only if every principal right ideal* I *of* R *has a supplement which is a direct summand.*

Proof. Since R is a semiregular ring, for all $a \in R$, R has a decomposition $R = A \oplus B$ with $A \subseteq aR$ and $B \cap aR \ll R$. Thus $R = aR + B$ and so B is a supplement of aR which is a summand of R. Conversely, let $R = aR + B$, $aR \cap B \ll B$ and B is a direct summand of R. Hence there exists $A \subseteq aR$ with $R = A \oplus B$ and so R is semiregular. □

Theorem 3.3. *Let* R *be a ring. The following are equivalent:*
(1) R *is semiregular.*
(2) *For every principal right ideal* I *of* R, *there exists an idempotent* $e^2 = e \in I$ *such that* $I\theta eR$.
(3) *For every principal left ideal* I *of* R, *there exists an idempotent* $e^2 = e \in I$ *such that* $I\theta Re$.
(4) *Every principal right ideal* I *of* R *has a supplement which is a direct summand.*
(5) *Every principal left ideal* I *of* R *has a supplement which is a direct summand.*

Proof. (1) ⇒ (2) Let $a \in R$. From (1), there exists an idempotent $e^2 = e \in aR$ such that $(1-e)a \in J(R)$. Since $eR \subseteq aR$, $\frac{aR+eR}{aR} \subseteq \frac{J(R)+aR}{aR}$. By modularity, $aR = eR \oplus ((1-e)R \cap aR)$. Hence $\frac{aR+eR}{eR} = \frac{eR\oplus((1-e)R\cap aR)}{eR} \subseteq \frac{eR+J(R)}{eR}$. Therefore $aR\theta eR$.

(2) ⇒ (4) Let $a \in R$. By (2), there exists an idempotent $e^2 = e \in R$ such that $aR\theta eR$. Since $R = eR \oplus (1-e)R$, $(1-e)R$ is a supplement of eR. Hence $(1-e)R$ is a supplement of aR in R by Theorem 2.5.

(4) ⇔ (1) By Lemma 3.2.

(1) ⇒ (3) ⇒ (5) ⇒ (1) It can be proved similarly. □

Theorem 3.4. *Let* R *be a ring. Then* R *is semiregular if and only if for each principal right ideal* I *of* R, *there exists a direct summand* A *and a small right ideal* H *of* R *such that* $I + H = A + H = I + A$.

Proof. Assume that R is semiregular. Then there exists a direct summand A of R such that $I\theta A$ by Theorem 3.3. By Proposition 2.4, $I\theta(I + A)$ and $A\theta(I + A)$. Note that $R = A \oplus A'$ for some $A' \leq R$. By Theorem 2.5, A' is a supplement for A, I and $I + A$. Hence $I + H = A + H = I + A$, where $H = (I + A) \cap A' \ll R$. The converse follows from Corollary 2.3. □

4. ν-Semiregular ring

We say that a ring R is *ν-semiregular* if for every semisimple principal right ideal aR of R there exists $e^2 = e \in R$ such that $(1-e)a \in J(R)$.

Clearly semiregular rings are ν-semiregular but the converse need not be true as we see in the following example.

Example 4.1. Let $\mathbb{Z}$ be the ring of all integers. Since $Soc(\mathbb{Z}) = 0$, $\mathbb{Z}$ is ν-semiregular. But $\mathbb{Z}$ is not semiregular since $n\mathbb{Z}$ has no supplement in $\mathbb{Z}$, for any $n \geq 2$.

Lemma 4.1. *Let $R = A + B$ where B is a right ideal of R and A is a semisimple right ideal of R. Then $R = A' \oplus B$ for some right ideal A' of A.*

Proof. Let A be a semisimple right ideal of R. Then $A \cap B$ is direct summand in A. So there exists right ideal A' of A such that $A = (A \cap B) \oplus A'$. Since $R = A + B$, then we obtain $R = ((A \cap B) \oplus A') + B = A' + B$. Thus $R = A' \oplus B$ because $(A \cap B) \cap A' = A' \cap B = 0$. □

Theorem 4.2. *Let R be a ring. Then the following are equivalent:*

(1) *R is ν-semiregular.*

(2) *Every semisimple principal right ideal of R has a supplement that is a direct summand.*

(3) *Every semisimple principal right ideal of R has a weak supplement.*

(4) *For every semisimple principal right ideal A of R there exists right ideal I of R such that $R = A + I$ and $A \cap I \subseteq J(I)$.*

Proof. (1) $\Leftrightarrow$ (2) It is similar to the proof of Lemma 3.2.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (4) Let A be a semisimple principal right ideal of R. Since A has a weak supplement, then there exists a right ideal I of R such that $A + I = R$ and $A \cap I \ll R$. By Lemma 4.1, $R = A' \oplus I$ for some right ideal A' of A. Then $A \cap I \ll R$ and so $A \cap I \subseteq J(I)$.

(4) $\Rightarrow$ (2) Let A be a semisimple principal right ideal of R. By hypothesis, there exists a right ideal H of R such that $R = A + H$ and $A \cap H \subseteq J(H)$. Thus $A \cap H \subseteq J(R)$ and so $A \cap H \ll R$. Since A is semisimple, by Lemma 4.1, $R = A' \oplus H$ for some right ideal A' of A. Thus we obtain $A \cap H \ll H$. □

Proposition 4.3. *Let R be a ν-semiregular ring. Then, for every $e^2 = e \in R$, eR is ν-semiregular.*

Proof. Let aR be a semisimple principal right ideal of eR. If $a = 0$, then eR is trivially ν-semiregular. Let $a \neq 0$, then $R = aR + I$ and $aR \cap I \ll I$ for some right ideal I of R. Then $eR = aR + (eR \cap I)$ and consequently by Lemma 4.1, $eR = A \oplus (eR \cap I)$ for some $A \subseteq aR$. Hence $eR \cap I$ is a direct summand of eR. Since $aR \cap I \ll I$, we have $aR \cap I \ll R$ and so $aR \cap I \ll eR$. Thus $aR \cap (eR \cap I) \ll eR \cap I$. Therefore $eR \cap I$ is a supplement of aR in eR. □

A *right distributive ring* is a ring whose lattice of right ideals is distributive.

Theorem 4.4. *Let $R = R_1 \oplus R_2$ be a right distributive ring. Then R is ν-semiregular if and only if each R_i is ν-semiregular.*

Proof. Let aR be a semisimple principal right ideal of R. Since R is right distributive, $aR = ((aR) \cap R_1) \oplus ((aR) \cap R_2)$. Let $a = a_1 + a_2$ where $a_1 \in R_1$ and $a_2 \in R_2$. Then $a_1R = aR \cap R_1$ and $a_2R = aR \cap R_2$. It is clear that a_1R and a_2R are semisimple. Thus there exists $A_i \leq R_i$ such that $R_i = a_iR + A_i$ and $(a_iR) \cap A_i \ll A_i$, for each $i = 1, 2$. Then $R = a_1R + a_2R + A_1 + A_2 = aR + A_1 + A_2$. Now we prove $aR \cap (A_1 + A_2) \ll A_1 + A_2$. Note that

$$\begin{aligned} aR \cap (A_1 + A_2) &= (aR \cap R_1 + aR \cap R_2) \cap (A_1 + A_2) \\ &\leq (A_1 \cap ((aR \cap R_1) + R_2)) + (A_2 \cap ((aR \cap R_2) + R_2)) \\ &\leq (aR \cap R_1) \cap (A_1 + R_2) + (aR \cap R_2) \cap (A_2 + R_1). \end{aligned}$$

On the other hand, $(aR \cap R_1) \cap (A_1 + R_2) = (a_1R) \cap (A_1 + R_2) \leq A_1 \cap (a_1R + R_2) \leq a_1R \cap (A_1 + R_2)$ implies that $a_1R \cap (A_1 + R_2) = A_1 \cap (a_1R + R_2) = (a_1R) \cap A_1$. Similarly, $a_2R \cap (A_2 + R_1) = A_2 \cap (a_2R + R_1) = (a_2R) \cap A_2$. Since $a_iR \cap A_i \ll A_i$, $a_1R \cap A_1 + a_2R \cap A_2 \ll A_1 + A_2$. Therefore $aR \cap (A_1 + A_2) \ll A_1 + A_2$. The converse is clear by Proposition 4.3. □

Proposition 4.5. *Let I be a small right ideal of a ring R and R/I a ν-semiregular ring. Then R is ν-semiregular.*

Proof. Let A be a semisimple principal right ideal of R. Then $\frac{A+I}{I}$ is a semisimple principal right ideal of $\frac{R}{I}$. If $\frac{R}{I} = \frac{A+I}{I}$, then $R = A + I$ and so $R = A$. Thus R is ν-semiregular. Let $\frac{A+I}{I}$ be a proper right ideal of $\frac{R}{I}$. By hypothesis, $\frac{A+I}{I}$ has a supplement $\frac{B}{I}$ in $\frac{R}{I}$. That is, $\frac{R}{I} = \frac{A+I}{I} + \frac{B}{I}$ and $\frac{A+I}{I} \cap \frac{B}{I} \ll \frac{B}{I}$. Therefore $R = A + B$ and $\frac{(A \cap B)+I}{I} \ll \frac{B}{I}$. By Lemma 4.1, $R = A' \oplus B$ for some right ideal A' of A. Now, we show that $A \cap B \ll B$: Let $B = (A \cap B) + X$ for some right ideal X of B. Then $\frac{B}{I} = \frac{(A \cap B)+I}{I} + \frac{X+I}{I}$. Since $\frac{(A \cap B)+I}{I} \ll \frac{B}{I}$, then $\frac{B}{I} = \frac{X+I}{I}$. It follows that $B = X + I$. As B is a direct summand of R, $B = X$. □

Lemma 4.6. *Let R be a ring with $Soc(R) \subseteq J(R)$. Then R is ν-semiregular.*

Proof. Clearly, if $Soc(R) = 0$, then R is ν-semiregular. Let aR be a semisimple principal right ideal of R, then $aR \subseteq Soc(R)$, so $aR \subseteq J(R) \ll R$. Then $R = R + aR$ and $aR \cap R = aR \ll R$. □

Corollary 4.7. *Let R be a right distributive ring, then R is ν-semiregular if and only if $Soc(R)$ has a supplement in R.*

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) Let A be a supplement of $Soc(R)$ in R. Then $R = Soc(R) + A$ and $Soc(A) = Soc(R) \cap A \ll A$. Hence, by Lemma 4.6, A is ν-semiregular. By Lemma 4.1, $R = B \oplus A$ where B is a semisimple right ideal of R. Hence, by Lemma 4.4, R is ν-semiregular. □

Corollary 4.8. *Let R be a ring. If $J(R) = R$, then R is ν-semiregular.*

Proof. Let $J(R) = R$, then $Soc(R) = Soc(J(R)) \ll R$. Then, by Lemma 4.6, R is ν-semiregular. □

Note that if R is a semisimple ring, then R is ν-semiregular if and only if R is semiregular. For another case, we give the following theorem which is the relation between ν-semiregular rings and semiregular rings.

Theorem 4.9. *Let R be a ring. Suppose that for any $a \in R$, there exists a semisimple principal right ideal I of R such that either $aR = I + T$ or $I = aR + T'$ for some right ideals $T, T' \ll R$. Then R is semiregular if and only if R is ν-semiregular.*

Proof. Suppose that R is a ν-semiregular ring and $a \in R$.

Case (1): Assume that there exists a semisimple principal right ideal I such that $aR = I + T$ for some small right ideal T of R. Then, by [4, 41.1(4)], R is semiregular.

Case (2): Assume that there exists a semisimple principal right ideal I of R such that $I = aR + T'$ for some small right ideal T' of R. Since R is a ν-semiregular ring, there exists a right ideal H of R such that $R = H + I$ and $H \cap I \ll H$. Hence $R = H + aR + T'$, and so $R = H + aR$ as $T' \ll R$. Note that $H \cap aR \leq H \cap I \ll H$. Thus R is semiregular. The converse is clear. □

Note that rings that every principal right ideal of them is semisimple, satisfy the condition of Theorem 4.9 (for example, $\mathbb{Z}_p$ satisfies the condition of Theorem 4.9 for any prime p).

References

[1] T. Amouzegar, On closed semiregular rings, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat.* **1** (2016), no. 2, 423–428.

[2] Y.X. Feng, L.Z. Kui, A generalization of semiregular rings, *J. Math. Research and Exposition* (6) (2009), no. 29, 1124–1130.

[3] W.K. Nicholson, Semiregular modules and rings, *Canad. J. Math.* **28** (1976), 1105–1120.

[4] R. Wisbauer, Foundations of module and ring theory, Gordon and Breach, Reading, 1991.

[5] G. Xiao, W. Tong, Generalizations of semiregular rings, *Comm. Algebra* **33** (10) (2005), 3447–3465.

[6] M.F. Yousif, Y. Zhou, Semiregular, semiperfect and perfect rings relative to an ideal, *Rocky Mountain J. Math.* **4** (2002), no. 32, 1651–1671.

[7] Y. Zhou, Generalizations of perfect, semiperfect and semiregular rings, *Algebra Colloq.* **3** (2000), no. 7, 305–18.

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A topological duality for M_3-lattices

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Abstract. In this article we determine a topological duality for M_3-lattices, introduced by A. V. Figallo in the journal Rev. Colombiana de Matemática, XXI, 1987 ([3]). By means of this duality we describe the congruences and the subdirectly irreducible M_3-lattices and reach some of Figallo's results in a different way.

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1. Introduction

In this work, we extend the duality obtained by H. A. Priestley for bounded distributive lattices (see [8] and [9]), known as *Priestley duality*, to the case of bounded M_3-lattices, showing that there exists a duality between the category whose objects are the bounded M_3-lattices and whose morphisms are the homomorphisms in the variety of the bounded M_3-lattices, and the category of M_3-spaces and M_3-functions.

By means of this duality we have managed to characterize the congruence lattice of an M_3-lattice in terms of certain closed subsets of its associated M_3-space, showing that there is an isomorphism between the lattice of the congruences and the dual lattice of certain closed subsets of its associated Priestley space, more precisely the closed and $\triangle$-involutive subsets.

Given that any variety of algebras is determined by its subdirectly irreducible algebras and what Birkhoff's Theorem states, that *Every non-trivial algebra A is isomorphic to a subdirect product of subdirectly irreducible algebras, each of which is a homomorphic image of A*, it is important to have their characterization. In this work we determine the simple and subdirectly irreducible M_3-lattices by using the characterization of the congruence lattice obtained and reach the same results as those achieved by Figallo in an algebraic way.

This article has been organized as follows. In Section 2 we introduce the definition and properties of M_3-lattices given by Figallo as well as some basic definitions of Priestley's duality. In Section 3 we describe a duality for M_3-lattices, starting with a study of the properties of M_3-lattice prime spectrum, which later allowed us to define the category of M_3-spaces and M_3-morphisms. Section 4 is devoted to the study of congruences and the determination of the simple and the subdirectly irreducible algebras, concluding that these algebras coincide, for which reason the variety is semisimple.

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