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(a): $\varphi(x,\cdot)$ is an N-function, *i.e.* convex, nondecreasing, continuous, $\varphi(x,0)=0$, $\varphi(x,t)>0$ for all $t>0$, and :

$$\lim_{t\to 0}\sup_{x\in\Omega}\frac{\varphi(x,t)}{t}=0 \quad , \quad \lim_{t\to\infty}\inf_{x\in\Omega}\frac{\varphi(x,t)}{t}=\infty,$$

(b): $\varphi(\cdot,t)$ is a measurable function.

A function $\varphi(x,t)$ which satisfies conditions (a) and (b) is called a Musielak-Orlicz function.

For every Musielak-Orlicz function $\varphi(x,t)$, we set $\varphi_x(t)=\varphi(x,t)$ and let $\varphi_x^{-1}(t)$ the reciprocal function with respect to t of $\varphi_x(t)$, i.e.

$$\varphi_x^{-1}(\varphi(x,t))=\varphi(x,\varphi_x^{-1}(t))=t.$$

For any two Musielak-Orlicz functions $\varphi(x,t)$ and $\gamma(x,t)$, we introduce the following ordering:

(c): If there exist two positive constants c and T such that for almost everywhere $x\in\Omega$:

$$\varphi(x,t)\leq\gamma(x,ct) \quad \text{for} \quad t\geq T,$$

we write $\varphi\prec\gamma$, and we say that γ dominate φ globally if $T=0$, and near infinity if $T>0$.

(d): For every positive constant c and almost everywhere $x\in\Omega$, if

$$\lim_{t\to 0}(\sup_{x\in\Omega}\frac{\varphi(x,ct)}{\gamma(x,t)})=0 \quad or \quad \lim_{t\to\infty}(\sup_{x\in\Omega}\frac{\varphi(x,ct)}{\gamma(x,t)})=0,$$

we write $\varphi\prec\prec\gamma$ at 0 or near ∞ respectively, and we say that φ increases essentially more slowly than γ at 0 or near ∞ respectively.

The Musielak-Orlicz function $\psi(x,t)$ complementary to (or conjugate of) $\varphi(x,t)$, in the sense of Young with respect to the variable t, is given by

$$\psi(x,s)=\sup_{t\geq 0}\{st-\varphi(x,t)\}, \tag{5}$$

and we have

$$st\leq\psi(x,s)+\varphi(x,t) \qquad \forall s,t\in I\!R^+. \tag{6}$$

The Musielak-Orlicz function $\varphi(x,t)$ is said to satisfy the Δ_2-condition if, there exists $k>0$ and a nonnegative function $h(\cdot)\in L^1(\Omega)$, such that

$$\varphi(x,2t)\leq k\varphi(x,t)+h(x) \qquad \text{a.e.} \quad x\in\Omega,$$

for large values of t, or for all values of t.

2.2. Musielak-Orlicz Lebesgue spaces. In this paper, the measurability of a function $u:\Omega\mapsto I\!R$ means the Lebesgue measurability.

We define the functional

$$\varrho_{\varphi,\Omega}(u)=\int_\Omega\varphi(x,|u(x)|)\,dx,$$

where $u:\Omega\mapsto I\!R$ is a measurable function. The set

$$K_\varphi(\Omega)=\{u:\Omega\longmapsto I\!R \quad \text{measurable} \;\; / \;\; \varrho_{\varphi,\Omega}(u)<+\infty\}$$

is called the Musielak-Orlicz class (or the generalized Orlicz class). The Musielak-Orlicz spaces (or the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$; equivalently

$$L_\varphi(\Omega) = \Big\{ u : \Omega \longmapsto I\!R \quad \text{measurable} \quad / \quad \varrho_{\varphi,\Omega}(\frac{|u(x)|}{\lambda}) < +\infty, \quad \text{for some } \lambda > 0 \Big\}.$$

In the space $L_\varphi(\Omega)$, we define the following two norms:

$$||u||_{\varphi,\Omega} = \inf \Big\{ \lambda > 0 \ \ / \ \ \int_\Omega \varphi(x, \frac{|u(x)|}{\lambda})\, dx \leq 1 \Big\},$$

which is called the Luxemburg norm, and the so-called Orlicz norm is given by:

$$|||u|||_{\varphi,\Omega} = \sup_{||v||_{\psi,\Omega} \leq 1} \int_\Omega |u(x)v(x)|\, dx,$$

where $\psi(x,t)$ is the Musielak-Orlicz function complementary (or conjugate) to $\varphi(x,t)$. These two norms are equivalent on the Musielak-Orlicz space $L_\varphi(\Omega)$.

The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. It is a separable space and $(E_\varphi(\Omega))^* = L_\psi(\Omega)$.

We have $E_\varphi(\Omega) = K_\varphi(\Omega)$ if and only if $K_\varphi(\Omega) = L_\varphi(\Omega)$ if and only if $\varphi(x,t)$ has the Δ_2-condition for large values of t, or for all values of t.

2.3. Musielak-Orlicz-Sobolev spaces. We now turn to the Musielak-Orlicz-Sobolev space $W^1L_\varphi(\Omega)$ (resp. $W^1E_\varphi(\Omega)$) is the space of all measurable functions u such that u and its distributional derivatives up to order 1 lie in $L_\varphi(\Omega)$ (resp. $E_\varphi(\Omega)$). Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with nonnegative integers α_i, $|\alpha| = |\alpha_1| + |\alpha_2| + \ldots + |\alpha_n|$ and $D^\alpha u$ denotes the distributional derivatives.
We define the convex modular and the norm on the Musielak-Orlicz-Sobolev spaces $W^1L_\varphi(\Omega)$ respectively by,

$$\overline{\varrho}_{\varphi,\Omega}(u) = \sum_{|\alpha| \leq 1} \varrho_{\varphi,\Omega}(D^\alpha u) \quad \textit{and} \quad ||u||_{1,\varphi,\Omega} = \inf \Big\{ \lambda > 0 : \overline{\varrho}_{\varphi,\Omega}(\frac{u}{\lambda}) \leq 1 \Big\},$$

for any $u \in W^1L_\varphi(\Omega)$.
The pair $\langle W^1L_\varphi(\Omega), ||u||_{1,\varphi,\Omega} \rangle$ is a Banach space if φ satisfies the following condition

$$\text{there exists a constant} \quad c > 0 \quad \text{such that} \quad \inf_{x \in \Omega} \varphi(x,1) \geq c.$$

The spaces $W^1L_\varphi(\Omega)$ and $W^1E_\varphi(\Omega)$ can be identified with subspaces of the product of $n+1$ copies of $L_\varphi(\Omega)$. Denoting this product by $\Pi L_\varphi(\Omega)$, we will use the weak topologies $\sigma(\Pi L_\varphi(\Omega), \Pi E_\psi(\Omega))$ and $\sigma(\Pi E_\psi(\Omega), \Pi L_\varphi(\Omega))$.

The space $W_0^1E_\varphi(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathrm{D}(\Omega)$ in $W^1E_\varphi(\Omega)$, and the space $W_0^1L_\varphi(\Omega)$ as the $\sigma(\Pi L_\varphi(\Omega), \Pi E_\psi(\Omega))$ closure of $\mathrm{D}(\Omega)$ in $W^1L_\varphi(\Omega)$, (for more details on Musielak-Orlicz-Sobolev spaces we refer to [24]).

2.4. Dual space. Let $W^{-1}L_\psi(\Omega)$ (resp. $W^{-1}E_\psi(\Omega)$) denotes the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_\psi(\Omega)$ (resp. $E_\psi(\Omega)$). It is a Banach space under the usual quotient norm.

If $\psi(x,t)$ has the Δ_2-condition, then the space $\mathrm{D}(\Omega)$ is dense in $W_0^1L_\varphi(\Omega)$ for the topology $\sigma(\Pi L_\varphi(\Omega), \Pi L_\psi(\Omega))$ (see corollary 1 of [9]).

3. Essential assumptions

Let Ω be a bounded open subset of $I\!R^N$ $(N \geq 2)$ with smooth boundary conditions. Let $\varphi(x,t)$ be a Musielak-Orlicz function and $\psi(x,t)$ the Musielak-Orlicz function complementary (or conjugate) to $\varphi(x,t)$. We assume here that $\psi(x,t)$ satisfying the Δ_2-condition near infinity, therefore $L_\psi(\Omega) = E_\psi(\Omega)$.
We assume that there exists an Orlicz function $M(t)$ such that $M(t) \prec \varphi(x,t)$ near infinity, i.e. there exist two constants $c > 0$ and $T \geq 0$ such that

$$M(t) \leq \varphi(x,ct) \qquad \text{a.e. in} \quad \Omega \qquad \text{for} \quad t \geq T. \tag{7}$$

Let $\Psi(\cdot)$ be a measurable function on Ω, such that

$$\Psi^+(\cdot) \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega),$$

and we consider the convex set

$$K_\Psi = \Big\{ v \in W_0^1 L_\varphi(\Omega) \ \textit{such that} \ v \geq \Psi \ \textit{a.e. in} \ \Omega \Big\}.$$

The Leray-Lions operator $A : D(A) \subset W_0^1 L_\varphi(\Omega) \longmapsto W^{-1} L_\psi(\Omega)$ given by

$$A(u) = -\text{div } a(x, \nabla u)$$

where $a : \Omega \times I\!R^N \longmapsto I\!R$ is a *Carathéodory* function (measurable with respect to x in Ω for every ξ in $I\!R^N$, and continuous with respect to ξ in $I\!R^N$ for almost every x in Ω) which satisfies the following conditions

$$|a(x,\xi)| \leq \beta\big(K(x) + k_1 \psi_x^{-1}(\varphi(x, k_2|\xi|))\big), \tag{8}$$

$$\big(a(x,\xi) - a(x,\xi^*)\big) \cdot \big(\xi - \xi^*\big) > 0 \qquad \text{for} \quad \xi \neq \xi^*, \tag{9}$$

$$a(x,\xi) \cdot \xi \geq \alpha \ \varphi(x, |\xi|), \tag{10}$$

for a.e. $x \in \Omega$ and all $\xi \in I\!R^N$, where $K(x)$ is a nonnegative function lying in $E_\psi(\Omega)$ and $\alpha, \beta > 0$ and $k_1, k_2 \geq 0$.
We consider the quasilinear unilateral elliptic problem

$$\begin{cases} -\text{div}\, a(x, \nabla u) = f & \text{in} \ \ \Omega, \\ u = 0 & \text{in} \ \ \partial\Omega, \end{cases} \tag{11}$$

with $f \in L^1(\Omega)$. We study the existence of entropy solution in the Musielak-Orlicz-Sobolev spaces.

4. Some technical lemmas

Now, we present some lemmas useful in the proof of our main results.

Lemma 4.1. *(see* [20], *Theorem 13.47) Let* $(u_n)_n$ *be a sequence in* $L^1(\Omega)$ *and* $u \in L^1(\Omega)$ *such that*

(i): $u_n \to u$ *a.e. in* Ω,

(ii): $u_n \geq 0$ *and* $u \geq 0$ *a.e. in* Ω,

(iii): $\displaystyle\int_\Omega u_n \, dx \to \int_\Omega u \, dx,$

then $u_n \to u$ *in* $L^1(\Omega)$.

Lemma 4.2. *Assuming that* (8)−(10) *hold, and let* $(u_n)_n$ *be a sequence in* $W_0^1 L_\varphi(\Omega)$ *such that*

(i): *$u_n \rightharpoonup u$ weakly in $W_0^1L_\varphi(\Omega)$ for $\sigma(\Pi L_\varphi(\Omega), \Pi E_\psi(\Omega))$,*
(ii): *$(a(x,\nabla u_n))_n$ is bounded in $(L_\psi(\Omega))^N = (E_\psi(\Omega))^N$,*
(iii): *Let $\Omega_s = \{x \in \Omega, \quad |\nabla u| \le s\}$ and χ_s his characteristic function, with*

$$\int_\Omega (a(x,\nabla u_n) - a(x,\nabla u\chi_s))\cdot(\nabla u_n - \nabla u\chi_s)\,dx \longrightarrow 0 \quad as \quad n,s\to\infty, \tag{12}$$

then $\varphi(x,|\nabla u_n|) \longrightarrow \varphi(x,|\nabla u|)$ in $L^1(\Omega)$ for a subsequence.

Proof. Taking $s \ge r > 0$, we have :

$$\begin{aligned} 0 &\le \int_{\Omega_r}(a(x,\nabla u_n) - a(x,\nabla u))\cdot(\nabla u_n - \nabla u)\,dx \\ &\le \int_{\Omega_s}(a(x,\nabla u_n) - a(x,\nabla u))\cdot(\nabla u_n - \nabla u)\,dx \\ &= \int_{\Omega_s}(a(x,\nabla u_n) - a(x,\nabla u\chi_s))\cdot(\nabla u_n - \nabla u\chi_s)\,dx \\ &\le \int_{\Omega}(a(x,\nabla u_n) - a(x,\nabla u\chi_s))\cdot(\nabla u_n - \nabla u\chi_s)\,dx. \end{aligned} \tag{13}$$

thanks to (12), we obtain

$$\lim_{n\to\infty}\int_{\Omega_r}(a(x,\nabla u_n) - a(x,\nabla u))\cdot(\nabla u_n - \nabla u)\,dx = 0. \tag{14}$$

Using the same argument as in [15], we claim that,

$$\nabla u_n \longrightarrow \nabla u \qquad \text{a.e. in} \quad \Omega. \tag{15}$$

On the other hand, we have

$$\begin{aligned} \int_\Omega a(x,\nabla u_n)\cdot\nabla u_n\,dx &= \int_\Omega (a(x,\nabla u_n) - a(x,\nabla u\chi_s))\cdot(\nabla u_n - \nabla u\chi_s)\,dx \\ &+ \int_\Omega a(x,\nabla u\chi_s)\cdot(\nabla u_n - \nabla u\chi_s)\,dx + \int_\Omega a(x,\nabla u_n)\cdot\nabla u\chi_s\,dx. \end{aligned} \tag{16}$$

For the second term on the right-hand side of (16), having in mind that $\psi(x,s)$ verify Δ_2-condition, then $L_\psi(\Omega) = E_\psi(\Omega)$, and thanks to (8) we have $a(x,\nabla u\chi_s) \in (E_\psi(\Omega))^N$. Moreover, we have $\nabla u_n \rightharpoonup \nabla u$ weakly in $(L_\varphi(\Omega))^N$ for $\sigma(\Pi L_\varphi(\Omega), \Pi E_\psi(\Omega))$, then

$$\begin{aligned} \lim_{s,n\to\infty}\int_\Omega a(x,\nabla u\chi_s)\cdot(\nabla u_n - \nabla u\chi_s)\,dx &= \lim_{s\to\infty}\int_\Omega a(x,\nabla u\chi_s)\cdot(\nabla u - \nabla u\chi_s)\,dx \\ &= \lim_{s\to\infty}\int_{\Omega/\Omega_s} a(x,0)\cdot\nabla u\,dx = 0. \end{aligned} \tag{17}$$

Concerning the last term on the right-hand side of (16), since $(a(x,\nabla u_n))_n$ is bounded in $(E_\psi(\Omega))^N$ and using (15), we obtain

$$a(x,\nabla u_n) \rightharpoonup a(x,\nabla u) \quad \text{weakly in} \quad (E_\psi(\Omega))^N \quad \text{for} \quad \sigma(\Pi E_\psi(\Omega), \Pi L_\varphi(\Omega)),$$

which implies that

$$\begin{aligned} \lim_{s,n\to\infty}\int_\Omega a(x,\nabla u_n)\cdot\nabla u\chi_s\,dx &= \lim_{s\to\infty}\int_\Omega a(x,\nabla u)\cdot\nabla u\chi_s\,dx \\ &= \int_\Omega a(x,\nabla u)\cdot\nabla u\,dx. \end{aligned} \tag{18}$$

By combining (12) and (16) – (18), we conclude that

$$\int_\Omega a(x,\nabla u_n)\cdot\nabla u_n\,dx \longrightarrow \int_\Omega a(x,\nabla u)\cdot\nabla u\,dx \quad \text{as} \quad n\to\infty. \tag{19}$$

On the other hand, we have $\varphi(x,|\nabla u_n|)\geq 0$ and $\varphi(x,|\nabla u_n|)\to\varphi(x,|\nabla u|)$ a.e. in Ω, by using the Fatou's Lemma we obtain

$$\int_\Omega \varphi(x,|\nabla u|)\,dx \leq \liminf_{n\to\infty}\int_\Omega \varphi(x,|\nabla u_n|)\,dx. \tag{20}$$

Moreover, since $a(x,\nabla u_n)\cdot\nabla u_n-\alpha\varphi(x,|\nabla u_n|)\geq 0$ and

$$a(x,\nabla u_n)\cdot\nabla u_n-\alpha\varphi(x,|\nabla u_n|)\longrightarrow a(x,\nabla u)\cdot\nabla u-\alpha\varphi(x,|\nabla u|) \qquad \text{a.e. in} \quad \Omega,$$

Thanks to Fatou's Lemma, we get

$$\int_\Omega a(x,\nabla u)\cdot\nabla u-\alpha\varphi(x,|\nabla u|)\,dx \leq \liminf_{n\to\infty}\int_\Omega a(x,\nabla u_n)\cdot\nabla u_n-\alpha\varphi(x,|\nabla u_n|)\,dx,$$

using (19), we obtain

$$\int_\Omega \varphi(x,|\nabla u|)\,dx \geq \limsup_{n\to\infty}\int_\Omega \varphi(x,|\nabla u_n|)\,dx. \tag{21}$$

By combining (20) and (21), we deduce

$$\int_\Omega \varphi(x,|\nabla u_n|)\,dx \longrightarrow \int_\Omega \varphi(x,|\nabla u|)\,dx \qquad \text{as} \quad n\to\infty. \tag{22}$$

In view of Lemma 4.1, we conclude that

$$\varphi(x,|\nabla u_n|)\longrightarrow\varphi(x,|\nabla u|) \qquad \text{in} \quad L^1(\Omega), \tag{23}$$

which finishes our proof.

5. Main results

Let $k>0$, we define the truncation function $T_k(\cdot):I\!R\longmapsto I\!R$ by

$$T_k(s)=\begin{cases} s & \text{if} \quad |s|\leq k,\\ k\dfrac{s}{|s|} & \text{if} \quad |s|>k.\end{cases}$$

Definition 5.1. A measurable function u is called an entropy solution of the quasi-linear unilateral elliptic problem (11) if

$$\begin{cases} T_k(u)\in K_\Psi \qquad \text{for any} \quad k>\|\Psi^+\|_\infty,\\ \displaystyle\int_\Omega a(x,\nabla u)\cdot\nabla T_k(u-v)\,dx\leq\int_\Omega fT_k(u-v)\,dx \qquad \forall v\in K_\Psi\cap L^\infty(\Omega).\end{cases} \tag{24}$$

Theorem 5.1. *Assuming that* (7) – (10) *hold, and* $f\in L^1(\Omega)$, *Then, the problem* (11) *has a unique entropy solution.*

5.1. Existence of entropy solution.

Step 1 : Approximate problems. Let $(f_n)_{n\in I\!N} \in W^{-1}E_\psi(\Omega)\cap L^\infty(\Omega)$ be a sequence of smooth functions such that $f_n \to f$ in $L^1(\Omega)$ and $|f_n| \leq |f|$ (for example $f_n = T_n(f)$). We consider the approximate problem

$$(P_n)\begin{cases} u_n \in K_\Psi, \\ \displaystyle\int_\Omega a(x,\nabla u_n)\cdot\nabla(u_n - v)\,dx \leq \int_\Omega f_n(u_n - v)\,dx \quad \text{for any} \quad v \in K_\Psi \cap L^\infty(\Omega). \end{cases} \tag{25}$$

Let $X = K_\Psi$, we define the operator $A : X \longmapsto X^*$ by

$$\langle Au, v\rangle = \int_\Omega a(x,\nabla u)\cdot\nabla v\ dx \qquad \forall v \in K_\Psi.$$

Using (6), we have for any $u, v \in K_\Psi$,

$$\begin{aligned} \Big|\int_\Omega a(x,\nabla u)\cdot\nabla v\,dx\Big| &\leq \int_\Omega \beta\big(K(x) + k_1\psi_x^{-1}(\varphi(x,k_2|\nabla u|))\big)|\nabla v|\,dx \\ &\leq \beta\int_\Omega \psi(x,K(x))\,dx + \beta k_1\int_\Omega \varphi(x,k_2|\nabla u|)\,dx + \beta(1+k_1)\int_\Omega \varphi(x,|\nabla v|)\,dx. \end{aligned} \tag{26}$$

Lemma 5.2. *The operator A acted from $W_0^1L_\varphi(\Omega)$ in to $W^{-1}L_\psi(\Omega) = W^{-1}E_\psi(\Omega)$ is bounded and pseudo-monotone. Moreover,A is coercive in the following sense : there exists $v_0 \in K_\Psi$ such that*

$$\frac{\langle Av, v - v_0\rangle}{||v||_{1,\varphi,\Omega}} \longrightarrow \infty \qquad as \quad ||v||_{1,\varphi,\Omega} \to \infty \qquad for \quad v \in K_\Psi.$$

Proof of Lemma 5.2. In view of (26), the operator A is bounded. For the coercivity, let $\varepsilon > 0$, we have for $v_0 \in K_\Psi$ and any $v \in W_0^1L_\varphi(\Omega)$

$$\begin{aligned} \big|\langle Av, v_0\rangle\big| &\leq \int_\Omega \big|a(x,\nabla v)\big|\ \big|\nabla v_0\big|\,dx \leq \beta\int_\Omega (K(x) + k_1\psi_x^{-1}(\varphi(x,k_2|\nabla v|)))|\nabla v_0|\,dx \\ &\leq \beta\int_\Omega K(x)|\nabla v_0|\,dx + \beta k_1\varepsilon\int_\Omega \psi_x^{-1}(\varphi(x,k_2|\nabla v|))\frac{1}{\varepsilon}|\nabla v_0|\,dx \\ &\leq \beta\int_\Omega \psi(x,K(x))dx + \beta\int_\Omega \varphi(x,|\nabla v_0|)dx + \beta k_1\varepsilon\int_\Omega \varphi(x,k_2|\nabla v|)\,dx \\ &\quad + \beta k_1\varepsilon\int_\Omega \varphi(x,\frac{1}{\varepsilon}|\nabla v_0|)\,dx \\ &\leq c_\varepsilon\int_\Omega \varphi(x,|\nabla v|)dx + \beta(k_1\varepsilon + 1)\int_\Omega \varphi(x,(\frac{1}{\varepsilon}+1)|\nabla v_0|)\,dx + C_1, \end{aligned}$$

with c_ε is a constant depending on ε. By taking ε small enough such that $c_\varepsilon \leq \frac{\alpha}{2}$, we obtain

$$\langle Av, v_0\rangle \leq \frac{\alpha}{2}\int_\Omega \varphi(x,|\nabla v|)dx + \beta(k_1\varepsilon + 1)\int_\Omega \varphi(x,(\frac{1}{\varepsilon}+1)|\nabla v_0|)\,dx + C_1.$$

On the other hand, in view of (10), we have

$$\langle Av, v\rangle = \int_\Omega a(x,\nabla v)\cdot\nabla v\,dx \geq \alpha\int_\Omega \varphi(x,|\nabla v|)\,dx.$$

Therefore

$$\frac{\langle Av, v-v_0\rangle}{||v||_{1,\varphi,\Omega}} = \frac{\langle Av, v\rangle - \langle Av, v_0\rangle}{||v||_{1,\varphi,\Omega}}$$

$$\geq \frac{\alpha\int_\Omega \varphi(x,|\nabla v|)\,dx - \frac{\alpha}{2}\int_\Omega \varphi(x,|\nabla v|)dx - \beta(k_1\varepsilon+1)\int_\Omega \varphi(x,(\frac{1}{\varepsilon}+1)|\nabla v_0|)\,dx + C_1}{\|v\|_{1,\varphi,\Omega}}$$

$$= \frac{\frac{\alpha}{2}\int_\Omega \varphi(x,|\nabla v|)\,dx - \beta(k_1\varepsilon+1)\int_\Omega \varphi(x,(\frac{1}{\varepsilon}+1)|\nabla v_0|)\,dx + C_1}{\|v\|_{1,\varphi,\Omega}} \longrightarrow \infty$$

as $\|v\|_{1,\varphi,\Omega}$ goes to infinity.
It remains to show that A is pseudo-monotone. Let $(u_k)_k$ be a sequence in $W_0^1L_\varphi(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u \ in\ W_0^1L_\varphi(\Omega) \quad \text{for} \quad \sigma(\Pi L_\varphi(\Omega), \Pi E_\psi(\Omega)), \\ Au_k \rightharpoonup \chi \ in\ W^{-1}E_\psi(\Omega) \quad \text{for} \quad \sigma(\Pi E_\psi(\Omega), \Pi L_\varphi(\Omega)), \\ \limsup_{k\to\infty}\langle Au_k, u_k\rangle \leq \langle \chi, u\rangle. \end{cases} \tag{27}$$

We will prove that

$$\chi = Au \ and\ \langle Au_k, u_k\rangle \to \langle \chi, u\rangle \ as\ k\to\infty.$$

Firstly, since $W_0^1L_\varphi(\Omega) \hookrightarrow\hookrightarrow E_\varphi(\Omega)$, then $u_k \to u$ in $E_\varphi(\Omega)$ for a subsequence still denoted $(u_k)_k$.
As $(u_k)_k$ is a bounded sequence in $W_0^1L_\varphi(\Omega)$ and thanks to the growth condition (8), it follows that $(a(x,\nabla u_k))_k$ is bounded in $(E_\psi(\Omega))^N$. Therefore, there exists a function $\xi \in (E_\psi(\Omega))^N$ such that

$$a(x,\nabla u_k) \rightharpoonup \xi \quad \text{in} \quad (E_\psi(\Omega))^N \quad \text{for} \quad \sigma(\Pi E_\psi(\Omega), \Pi L_\varphi(\Omega)) \quad \text{as} \quad k\to\infty. \tag{28}$$

It is clear that, for all $v \in W_0^1L_\varphi(\Omega)$, we have

$$\langle \chi, v\rangle = \lim_{k\to\infty}\langle Au_k, v\rangle = \lim_{k\to\infty}\int_\Omega a(x,\nabla u_k)\cdot\nabla v\,dx = \int_\Omega \xi\cdot\nabla v\,dx. \tag{29}$$

By using (27) and (29), we obtain

$$\limsup_{k\to\infty}\langle Au_k, u_k\rangle = \limsup_{k\to\infty}\int_\Omega a(x,\nabla u_k)\cdot\nabla u_k\,dx \leq \int_\Omega \xi\cdot\nabla u\,dx. \tag{30}$$

On the other hand, thanks to (9), we have

$$\int_\Omega \Big(a(x,\nabla u_k) - a(x,\nabla u)\Big)\cdot(\nabla u_k - \nabla u)\,dx \geq 0, \tag{31}$$

then

$$\int_\Omega a(x,\nabla u_k)\cdot\nabla u_k\,dx \geq \int_\Omega a(x,\nabla u_k)\cdot\nabla u\,dx + \int_\Omega a(x,\nabla u)\cdot(\nabla u_k - \nabla u)\,dx.$$

In view of (28), we have

$$\liminf_{k\to\infty}\int_\Omega a(x,\nabla u_k)\cdot\nabla u_k\,dx \geq \int_\Omega \xi\cdot\nabla u\,dx$$

and (30) yields

$$\lim_{k\to\infty}\int_\Omega a(x,\nabla u_k)\cdot\nabla u_k\,dx=\int_\Omega \xi\cdot\nabla u\,dx. \tag{32}$$

Combining (29) and (32), we find:

$$\langle Au_k,u_k\rangle\to\langle\chi,u\rangle \qquad \text{as} \quad k\to\infty. \tag{33}$$

In view of (32), we have

$$\lim_{k\to\infty}\int_\Omega \Big(a(x,\nabla u_k)-a(x,\nabla u)\Big)\cdot(\nabla u_k-\nabla u)\,dx\to\ 0$$

which implies, thanks to Lemma 4.2, that

$$u_k\to u \quad \text{in} \quad W_0^1L_\varphi(\Omega) \qquad \text{and} \qquad \nabla u_k\to\nabla u \quad \text{a.e. in} \quad \Omega,$$

then

$$a(x,\nabla u_k)\rightharpoonup a(x,\nabla u) \quad \text{in} \quad (E_\psi(\Omega))^N,$$

we deduce that $\chi=Au$, which completes the proof the Lemma 5.2. □

In view of Lemma 5.2, there exists at least one weak solution $u_n\in W_0^1L_\varphi(\Omega)$ of the problem (25), (cf. [10], Lemma 6).

Step 2 : A priori estimates. Taking $v=u_n-\eta T_k(u_n-\Psi^+)\in W_0^1L_\varphi(\Omega),$ for η small enough we have $v\geq\Psi,$ thus v is an admissible test function in (25), and we obtain

$$\int_\Omega a(x,\nabla u_n)\cdot\nabla T_k(u_n-\Psi^+)\,dx\leq\int_\Omega f_nT_k(u_n-\Psi^+)\,dx,$$

Since $\nabla T_k(u_n-\Psi^+)$ is identically zero on the set $\{|u_n-\Psi^+|>k\}$, we can write

$$\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\nabla(u_n-\Psi^+)\ dx\leq\int_\Omega f_nT_k(u_n-\Psi^+)\,dx\leq C_2k,$$

with $C_2=\|f\|_1,$ it follows that

$$\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\nabla u_n\,dx\leq C_2k+\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\nabla\Psi^+\,dx.$$

Let $0<\lambda<\dfrac{\alpha}{\alpha+1}$, it's clear that

$$\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\nabla u_n\,dx\leq C_2k+\lambda\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\frac{\nabla\Psi^+}{\lambda}\,dx. \tag{34}$$

Thanks to (9), we have

$$\int_{\{|u_n-\Psi^+|\leq k\}}\Big(a(x,\nabla u_n)-a(x,\frac{\nabla\Psi^+}{\lambda})\Big)\cdot(\nabla u_n-\frac{\nabla\Psi^+}{\lambda})\,dx\geq 0,$$

then

$$\begin{aligned}\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\frac{\nabla\Psi^+}{\lambda}\,dx \quad &\leq\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\nabla u_n\,dx\\ &-\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\frac{\nabla\Psi^+}{\lambda})\cdot(\nabla u_n-\frac{\nabla\Psi^+}{\lambda})\,dx.\end{aligned}$$